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Exact scheme independence at one loop

Stefano Arnone, Antonio Gatti and Tim R. Morris

Department of Physics and Astronomy, University of Southampton
Highfield, Southampton SO17 1BJ, U.K.
E-mails: sa@hep.phys.soton.ac.uk, gatti@hep.phys.soton.ac.uk, T.R.Morris@soton.ac.uk

Abstract: The requirement that the quantum partition function be invariant under a renormalization group transformation results in a wide class of exact renormalization group equations, differing in the form of the kernel. Physical quantities should not be sensitive to the particular choice of the kernel. We demonstrate this scheme independence in four dimensional scalar field theory by showing that, even with a general kernel, the one-loop beta function may be expressed only in terms of the effective action vertices, and thus, under very general conditions, the universal result is recovered.

Keywords: Renormalization Regularization and Renormalons, Nonperturbative Effects.

Contents

1. Introduction

The non-perturbative meaning of renormalization, as understood by Wilson, is formulated most directly in the continuum in terms of the exact Renormalization Group (RG) [1]. Moreover, the fact that solutions S of the corresponding flow equations can then be found directly in terms of renormalized quantities, that all physics (e.g. Green functions) can be extracted from this wilsonian effective action S, and that renormalizability is trivially preserved in almost any approximation [2, 3], turns these ideas into a powerful framework for considering non-perturbative analytic approximations (see for example refs. [1]–[5], [8], [10]–[14], [18]).

In the past a number of different versions and ways of deriving the exact RG have been proposed [1]–[10], but in fact the resulting flow equations may readily be shown to be equivalent under changes of variables [2, 3, 11, 12, 13].

Recently, far more general versions of the exact RG have been considered, dependent on the choice of a functional Ψ, known as the "kernel" of the exact RG [10, 13]. This elucidates the structure behind all forms of exact RG [13]. In particular, each exact RG is associated with a Ψ, that induces a reparametrization (field redefinition) along the flow, and acts as a connection between the theory space of actions at different scales Λ. As a result, local to some generic point Λ on the flow, all the exact RGs, including these generalised ones, may be shown to be just reparametrizations of each other. When this reparametrization can be extended globally, the result is an immediate proof of scheme independence for physical observables. Indeed computations of physical quantities then differ only through some field reparametrization. One practical example is an explicit field redefinition that interpolates between results computed using different choices of cutoff function $c(p^2/\Lambda^2)$ [13].

Even more dramatic than this however, is the use of this freedom to adapt the exact RG to certain forms of approximation or special physical problems [13]. In particular, recently there has been substantial progress in adapting these ideas to gauge theory. It turns out that not only can one introduce an effective cutoff Λ in a way that does not break the gauge invariance [15] but careful choices of Ψ allow the gauge invariance to be preserved manifestly (i.e. not even gauge fixed) along the flow and in the solutions S [10, 14, 16].

Nevertheless, the analysis presented in ref. [13] leaves open a number of practical questions. Although at general points of the flow, all exact RGs are locally equivalent, obstructions can arise to full (global) equivalence, on the one hand from differences in the global structure of fixed points deduced from the two flows, and on the other from the non-existence at special points of an inverse in an implied change of variables (from S to Λ) [13]. However it is difficult to see in general how one can determine when such obstructions do exist, given that in practice one has to make approximations in order solve these theories. Furthermore, computations within a generalised exact RG, such as the type being used for gauge theory [10, 14], generate many more terms, whose link to standard text-book methods of doing quantum field theory seem quite obscure. Computations of physical quantities (such as the large-N SU(N) Yang-Mills one-loop β function) fall out as the universal expected result but only after a complicated calculation and apparently by magic [10, 14].

This paper addresses the above problems within a sufficiently simple and bounded context: the one-loop β function of four dimensional (one component) scalar field theory. We will see that even for a very general form of Ψ (one involving a general 'seed' action $\hat{S}$), the correct universal result is obtained. To our knowledge, this is the first concrete test of such scheme independence beyond testing for cutoff function independence. The only requirements we have to impose on $\hat{S}$ to recover the universal result, are some very weak and general requirements which are necessary in any case to ensure that the wilsonian action S makes sense. To this level then, all such exact RGs are equivalent and merely parametrise changes of scheme.

From a practical point of view, we find that a major step in understanding and solving these equations comes from using the flow equations to exchange elements of $\hat{S}$ (more generally Ψ) in favour of vertices from the wilsonian action S. Some expressions for the quantum corrections in S then turn out to simplify dramatically and result in scheme 'covariant' expressions with straightforward physical interpretations. The one-loop β function, β_1, is one example where this covariance then implies even invariance under changes of scheme. Indeed, we find very directly in this way, the same answer for β_1 independent of the details of the cutoff function c and the seed action $\hat{S}$.

The plan of the paper is as follows. In section 2, we review briefly one version of the exact RG, Polchinski's equation [7], use this a basis to explain the more general exact RGs, and introduce our concrete set of such things, involving $\hat{S}$. In section 3, we perform the one-loop calculation using this general form of exact RG. Finally, in section 4, we summarise and draw our conclusions.

2. From the Polchinski equation to general exact RGs

We will consider a massless scalar field theory in four euclidean dimensions, with a momentum space cutoff. The theory is defined at some ultra-violet scale, Λ_0, by giving the quantum partition function,

$$\mathcal{Z}_{\Lambda_0} = \int \mathcal{D}[\varphi]\, \mathrm{e}^{-S_{\Lambda_0}[\varphi]} \,. \tag{2.1}$$

The action consists of the kinetic term, regularised by the introduction of a cutoff function, and the characteristic self-interaction term,

$$S_{\Lambda_0}[\varphi] = \frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\, p^2 c^{-1}\left(\frac{p^2}{\Lambda_0^2}\right)\varphi^2 + \frac{\lambda_0}{4!}\int d^4x\, \varphi^4 \,. \tag{2.2}$$

Here $c(p^2/\Lambda^2) > 0$ is a *smooth*, i.e. infinitely differentiable, ultra-violet cutoff profile. The cutoff, which modifies propagators $1/p^2$ to c/p^2, satisfies $c(0) = 1$ so that low energies are unaltered, and $c(p^2/\Lambda^2) \to 0$ as $p^2/\Lambda^2 \to \infty$ sufficiently fast that all Feynman diagrams are ultra-violet regulated.

The RG transformation amounts to changing the cutoff from Λ_0 to $\Lambda \ll \Lambda_0$ and compensating for such change by replacing the action with a more complicated effective action, a.k.a. the wilsonian action [1],

$$S[\varphi] = \frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\, p^2 c^{-1}\left(\frac{p^2}{\Lambda^2}\right)\varphi^2 + S^{\mathrm{int}}[\varphi;\Lambda] \,. \tag{2.3}$$

Demanding that physics be invariant under the RG transformation results in a functional differential equation for the effective interaction [7],

$$\Lambda\partial_\Lambda S^{\mathrm{int}} = -\frac{1}{\Lambda^2}\frac{\delta S^{\mathrm{int}}}{\delta\varphi}\cdot c'\cdot\frac{\delta S^{\mathrm{int}}}{\delta\varphi} + \frac{1}{\Lambda^2}\frac{\delta}{\delta\varphi}\cdot c'\cdot\frac{\delta S^{\mathrm{int}}}{\delta\varphi} \,. \tag{2.4}$$

In eq. (2.4), prime denotes differentiation with respect to the function's argument (here p^2/Λ^2) and the following shorthand has been introduced: for two functions $f(x)$ and $g(y)$ and a momentum space kernel $W(p^2/\Lambda^2)$, with Λ being the effective cutoff,

$$f\cdot W\cdot g = \int\!\!\int d^4x\, d^4y\, f(x) W_{xy} g(y) \,, \tag{2.5}$$

where $W_{xy} = \int \frac{d^4p}{(2\pi)^4} W(p^2/\Lambda^2)\mathrm{e}^{ip\cdot(x-y)}$.

Note that the regularised kinetic term in eq. (2.3) may be written as

$$\frac{1}{2}\,\partial_\mu\varphi\cdot c^{-1}\cdot\partial_\mu\varphi \,. \tag{2.6}$$

This will be referred to as the seed action and denoted by $\hat{S}$. In terms of the total effective action, $S[\varphi] = \hat{S} + S^{\mathrm{int}}$, and $\Sigma \doteq S - 2\hat{S}$, the exact RG equation reads

$$\Lambda\partial_\Lambda S = -\frac{1}{\Lambda^2}\frac{\delta S}{\delta\varphi}\cdot c'\cdot\frac{\delta\Sigma}{\delta\varphi} + \frac{1}{\Lambda^2}\frac{\delta}{\delta\varphi}\cdot c'\cdot\frac{\delta\Sigma}{\delta\varphi} \tag{2.7}$$

(up to a vacuum energy term that was discarded in [7]).

The invariance of the partition function is manifest if eq. (2.7) is recast as

$$\Lambda\partial_\Lambda \mathrm{e}^{-S} = -\frac{1}{\Lambda^2}\frac{\delta}{\delta\varphi}\cdot c'\cdot\left(\frac{\delta\Sigma}{\delta\varphi}\,\mathrm{e}^{-S}\right), \tag{2.8}$$

i.e. the infinitesimal RG transformation results in a change in the integrand which is a total functional derivative.

Incidentally, this estabilishes a rather counterintuitive result, that integrating out degrees of freedom is just equivalent to redefining the fields in the theory [14, 13]. In the present case, the change in the partition function may be shown to correspond to the change of variables $\varphi \to \varphi + \delta\Lambda\,\Psi$, with the "kernel" $\Psi = -\frac{1}{\Lambda^3}\,c'\,\frac{\delta\Sigma}{\delta\varphi}$ that appears in eq. (2.8) [13].

Different forms of exact RG equations correspond to choosing different kernels Ψ. There is a tremendous amount of freedom in this choice: more general seed actions $\hat{S}$ can be considered, c' can be augmented by further terms including interactions (analogous to the "wines" in refs. [10, 14]), higher functional derivatives and/or more complex dependences on the effective action S can be included [13].

Intuitively, we should expect that there is a wide freedom in the choice of kernel, just as there is a great deal of freedom in choosing the form of a blocking transformation in the condensed matter or lattice realisation of the wilsonian renormalization group [1]. The flow equation eq. (2.4) is distinguished only by its relative simplicity (related to incorporating the cutoff only in the kinetic term). Nevertheless, physical quantities should turn out to be universal, i.e. independent of these choices. It can be shown that local to generic points Λ, all of these choices are actually related to each other by field redefinitions [13]. If no obstructions exist to extending this for all Λ, then universality follows immediately. However obstructions can arise from the inability to invert these maps and from global restrictions (in particular arising from the structure of fixed points) [13]. In this paper we investigate these issues at a completely concrete level, computing the one-loop β function in scalar field theory with a generalised Ψ. With so much freedom we have to restrict it to be able to be concrete; we choose to consider a general seed action $\hat{S}$ of a form that we now outline.

As shown above, Polchinski's equation comes from setting the seed action equal to the effective kinetic term in the wilsonian effective action eq. (2.3). As we will show in the next section, if we are to reproduce at the classical level the form of the effective kinetic term in eq. (2.3), the bilinear term in $\hat{S}$ must continue to be equal to eq. (2.6).[1] Furthermore, we choose to leave the $\varphi \leftrightarrow -\varphi$ symmetry alone, by requiring that $\hat{S}$ is even under this symmetry. We are left with a generalised exact renormalization group parametrised by the infinite set of seed action n-point vertices, $n = 4, 6, 8, \ldots$. We will leave each of these vertices as completely unspecified functions of their momenta except for the requirement that *the vertices be infinitely differentiable and lead to convergent momentum integrals.* (The first condition ensures that no spurious infrared singularities are introduced and that all effective vertices can be Taylor expanded in their momenta to any order [10, 3]. The second condition is necessary for the flow equation to make sense at the quantum level and also ensures that the flow actually corresponds to integrating out modes [14, 13].),

We are therefore incorporating in the momentum dependence of *each* of the seed action n-point vertices, $n = 2, 4, 6, \ldots$, an infinite number of parameters. Of course these infinite number of vertices, each with an infinite number of parameters, then appear in the effective action S as a consequence of the flow equation. Remarkably, we can still compute the one-loop β function. Moreover, as we will see in the next section, we can invert the flow

[1] It is not necessary that the classical kinetic term take the form in eq. (2.3). We choose to require it purely for convenience.

equation by expressing $\hat{S}$ vertices in terms of S, and in this way demonstrate explicitly that the result is universal — viz. independent of the choice of c and $\hat{S}$. It follows that, at least in this case, the *only* requirements that we need to impose on $\hat{S}$ in order to ensure that these generalised flow equations continue to describe the same physics are those italicized in the previous paragraph.

3. One-loop beta function with general seed action

Physical quantities must be universal, i.e. independent of the renormalization scheme. In particular, they should not be sensitive to the particular choice of the RG kernel, e.g. on the form of the cutoff function or the expression for the seed action. We aim to calculate one of those, the one-loop contribution to the β function, while keeping as general a seed action as possible. As we will see, an elegant, clear cut way of achieving such a result is to make use of the equations of motion for the effective couplings in order to get rid of the seed action vertices.

As usual, universal results are obtained only after the imposition of appropriate renormalization conditions which allow us to define what we mean by the physical (more generally renormalized) coupling and field. (The renormalized mass must also be defined and is here set to zero implicitly by ensuring that the only scale that appears is Λ.)

We write the vertices of S as

$$S^{(2n)}(\vec{p};\Lambda) \equiv S^{(2n)}(p_1,p_2,\dots,p_{2n};\Lambda) \doteq (2\pi)^{8n}\frac{\delta^{2n}S}{\delta\varphi(p_1)\delta\varphi(p_2)\cdots\delta\varphi(p_{2n})}\,, \tag{3.1}$$

(and similarly for the vertices of $\hat{S}$). In common with earlier works [7, 9], we define the renormalized four-point coupling λ by the effective action's four-point vertex evaluated at zero momenta: $\lambda(\Lambda) = S^{(4)}(\vec{0};\Lambda)$. This makes sense once we express quantities in terms of the renormalized field, defined (as usual) to bring the kinetic term into canonical form $S^{(2)}(p,-p;\Lambda) = S^{(2)}(0,0;\Lambda) + p^2 + O(p^4/\Lambda^2)$. The flow equation can then be taken to be of the form [12, 17]:

$$\Lambda\partial_\Lambda S - \frac{\gamma}{2}\varphi\cdot\frac{\delta S}{\delta\varphi} = -\frac{1}{\Lambda^2}\frac{\delta S}{\delta\varphi}\cdot c'\cdot\frac{\delta\Sigma}{\delta\varphi} + \frac{1}{\Lambda^2}\frac{\delta}{\delta\varphi}\cdot c'\cdot\frac{\delta\Sigma}{\delta\varphi}\,. \tag{3.2}$$

We have used the short hand

$$\varphi\cdot\frac{\delta S}{\delta\varphi} \doteq \int\frac{d^4p}{(2\pi)^4}\varphi(p)\frac{\delta S}{\delta\varphi(p)} \tag{3.3}$$

and as usual the anomalous dimension $\gamma = \frac{1}{Z}\Lambda\partial_\Lambda Z$, where Z is the wavefunction renormalization. As emphasised in refs. [10, 13], although eq. (3.2) is not the result of changing variables $\varphi \mapsto \varphi\sqrt{Z}$ in eq. (2.7), it is still a perfectly valid flow equation and a more appropriate starting point when wavefunction renormalization has to be taken into account. This is in fact a small example of the immense freedom we have in defining the flow equation. (The new term on the left hand side arises from replacing $\partial_\Lambda|_\varphi$ with a partial derivative at constant renormalized field, but in order to produce the right hand side, and in order to reproduce the same $\hat{S}$, we need to start with the alternative cutoff function cZ in eqs. (2.2)–(2.7). Alternatively, for the purposes of computing the β function, we could have simply taken account of the wavefunction renormalization afterwards as in ref. [18].)

We now rescale the field φ to

$$\varphi = \frac{1}{\sqrt{\lambda}}\,\tilde{\varphi}\,, \tag{3.4}$$

so as to put the coupling constant in front of the action. This ensures the expansion in the coupling constant coincides with the one in $\hbar$, the actual expansion parameter being just $\lambda\hbar$. The resulting expansion is more elegant, being no longer tied at the same time to the order of expansion of the field φ. It is also analogous to the treatment pursued for gauge theory in refs. [10, 14] (where gauge invariance introduces further simplications in particular forcing $\gamma = 0$ for the new gauge field). The following analysis thus furnishes a demonstration that these ideas also work within scalar field theory.

The bare action (2.2) rescales as

$$S_{\Lambda_0}[\varphi] = \frac{1}{\lambda}\left[\frac{1}{2}\int\frac{d^4p}{(2\pi)^4}\,p^2c^{-1}(\frac{p^2}{\Lambda_0^2})\,\tilde{\varphi}^2 + \frac{1}{4!}\int d^4x\,\tilde{\varphi}^4\right] \doteq \frac{1}{\lambda}\tilde{S}_{\Lambda_0}[\tilde{\varphi}]\,. \tag{3.5}$$

Defining the “rescaled” effective and seed actions as $S[\varphi] = \frac{1}{\lambda}\tilde{S}[\tilde{\varphi}]$, $\hat{S}[\varphi] = \frac{1}{\lambda}\tilde{\hat{S}}[\tilde{\varphi}]$, and absorbing the change to $\partial_\Lambda|_{\tilde{\varphi}}$ in a change to $\tilde{\gamma}$, the flow equation (2.7) reads

$$\Lambda\partial_\Lambda\left(\frac{1}{\lambda}\tilde{S}\right) - \frac{\tilde{\gamma}}{2\lambda}\tilde{\varphi}\cdot\frac{\delta\tilde{S}}{\delta\tilde{\varphi}} = -\frac{1}{\lambda\Lambda^2}\frac{\delta(\tilde{S}-2\tilde{\hat{S}})}{\delta\tilde{\varphi}}\cdot c'\cdot\frac{\delta\tilde{S}}{\delta\tilde{\varphi}} + \frac{1}{\Lambda^2}\frac{\delta}{\delta\tilde{\varphi}}\cdot c'\cdot\frac{\delta(\tilde{S}-2\tilde{\hat{S}})}{\delta\tilde{\varphi}}\,. \tag{3.6}$$

Expanding the action, the beta function $\beta(\Lambda) = \Lambda\partial_\Lambda\lambda$ and anomalous dimension, in powers of the coupling constant:

$$\begin{aligned}\tilde{S}[\tilde{\varphi}] &= \tilde{S}_0 + \lambda\tilde{S}_1 + \lambda^2\tilde{S}_2 + \cdots\,,\\ \beta(\Lambda) &= \beta_1\lambda^2 + \beta_2\lambda^3 + \cdots\,,\\ \tilde{\gamma}(\Lambda) &= \tilde{\gamma}_1\lambda + \tilde{\gamma}_2\lambda^2 + \cdots\end{aligned}$$

yields the loopwise expansion of the flow equation[2]

$$\Lambda\partial_\Lambda S_0 = -\frac{1}{\Lambda^2}\frac{\delta S_0}{\delta\varphi}\cdot c'\cdot\frac{\delta(S_0-2\hat{S})}{\delta\varphi}\,, \tag{3.7}$$

$$\Lambda\partial_\Lambda S_1 - \beta_1 S_0 - \frac{\gamma_1}{2}\varphi\cdot\frac{\delta S_0}{\delta\varphi} = -\frac{2}{\Lambda^2}\frac{\delta S_1}{\delta\varphi}\cdot c'\cdot\frac{\delta(S_0-\hat{S})}{\delta\varphi} + \frac{1}{\Lambda^2}\frac{\delta}{\delta\varphi}\cdot c'\cdot\frac{\delta(S_0-2\hat{S})}{\delta\varphi}\,, \tag{3.8}$$

etc. γ_1 and β_1 may now be extracted directly from eq. (3.8), as specialised to the two-point and four-point effective couplings, $S^{(2)}(\vec{p};\Lambda)$ and $S^{(4)}(\vec{p};\Lambda)$ respectively, once the renormalization conditions have been taken into account.

We impose the wavefunction renormalization condition in the new variables:

$$S^{(2)}(p,-p;\Lambda) = S^{(2)}(0,0;\Lambda) + p^2 + O\left(\frac{p^4}{\Lambda^2}\right)\,. \tag{3.9}$$

[2]In order to simplify the notation, the tildes will be removed from now on.

Bearing in mind that the coupling constant has been scaled out, we impose the condition

$$S^{(4)}(\vec{0};\Lambda) = 1\,. \tag{3.10}$$

Both conditions eq. (3.9) and eq. (3.10) are already saturated at tree level. (To see this it is sufficient to note that, since the theory is massless, the only scale involved is Λ. Since $S_0^{(4)}$ is dimensionless it must be a constant at null momenta, thus $S_0^{(4)}(\vec{0};\Lambda) = S_0^{(4)}(\vec{0};\Lambda_0) = 1$. Similar arguments apply to $S_0^{(2)}$.) Hence the renormalization condition implies that we must have no quantum corrections to the four-point vertex at $\vec{p} = \vec{0}$, or to the $O(p^2)$ part of the two-point vertex, i.e.

$$S_n^{(4)}(\vec{0};\Lambda) = 0 \quad \text{and} \quad S_n^{(2)}(p,-p;\Lambda)\Big|_{p^2} = 0 \qquad \forall n \geq 1\,, \tag{3.11}$$

where the notation $|_{p^2}$ means that one should take the coefficient of p^2 in the series expansion in p. The flow equations for these special parts of the quantum corrections thus greatly simplify, reducing to algebraic equations which then determine the β_i and γ_i. In particular, from the flow of $S_1^{(4)}$ at null momenta:[3]

$$\beta_1 + 2\gamma_1 = \frac{8c_0'}{\Lambda^2}\left[1 - \hat{S}^{(4)}(\vec{0})\right] S_1^{(2)}(0) - \frac{1}{\Lambda^2}\int_q c'\left(\frac{q^2}{\Lambda^2}\right)\left[S_0^{(6)} - 2\hat{S}^{(6)}\right](\vec{0},q,-q)\,, \tag{3.12}$$

where $c_0' = c'(0)$ and $\int_q \doteq \int \frac{d^4q}{(2\pi)^4}$, and from the flow of $S_1^{(2)}$ expanded to $O(p^2)$:

$$\beta_1 + \gamma_1 = -\frac{1}{\Lambda^2}\int_q c'\left(\frac{q^2}{\Lambda^2}\right)\left[S_0^{(4)} - 2\hat{S}^{(4)}\right](p,-p,q,-q)\Big|_{p^2}\,. \tag{3.13}$$

Note that contrary to the standard text book derivation our one-loop anomalous dimension is not zero, picking up a contribution from the general field reparametrization [13] induced by higher point terms in $\hat{S}$ and a contribution $-\beta_1$ due to the field rescaling eq. (3.4).

In order to evaluate eq. (3.12), we need to calculate $S_1^{(2)}(0)$ and $S_0^{(6)}(\vec{0},q,-q)$. We would also need $\hat{S}^{(4)}(\vec{0})$ and $\hat{S}^{(6)}(\vec{0},q,-q)$, but we will see that we can avoid using explicit expressions for them, and thus keep $\hat{S}$ general, by using the equations of motion to express them in terms of the effective vertices $S_0^{(4)}$ and $S_0^{(6)}$.

However, as explained in the previous section, our $\hat{S}$ is not completely arbitrary. Apart from some very general requirements on the differentiability and integrability of its vertices, for convenience we restrict $\hat{S}$ to have only even-point vertices, as in fact already used in eqs. (3.12) and (3.13), and constrain its two-point vertex so that the two-point effective coupling keeps the same functional dependence upon Λ as the bare one (as in eq. (2.3)). This last condition reads

$$S_0^{(2)}(p) = p^2 c^{-1}\left(\frac{p^2}{\Lambda^2}\right) \tag{3.14}$$

and from the two-point part of eq. (3.7), we immediately find

$$\hat{S}^{(2)}(p) = p^2 c^{-1}\left(\frac{p^2}{\Lambda^2}\right). \tag{3.15}$$

[3]Here and later we suppress the Λ dependence of the S and $\hat{S}$ vertices.

Let us start with the calculation of $S_1^{(2)}(0)$. From eq. (3.8), its equation reads

$$\Lambda\partial_\Lambda S_1^{(2)}(0) = \frac{1}{\Lambda^2}\int_q c'(\frac{q^2}{\Lambda^2})\left[S_0^{(4)} - 2\hat{S}^{(4)}\right](0,0,q,-q)\,, \tag{3.16}$$

where eqs. (3.15) and (3.14) have been already used to cancel out the classical terms. Pursuing our strategy, we get rid of $\hat{S}^{(4)}$ by making use of the equation of motion for the four-point effective coupling at tree level

$$\Lambda\partial_\Lambda S_0^{(4)}(\vec{p}) = \frac{2}{\Lambda^2}\sum_i \frac{p_i^2 c'_{p_i}}{c_{p_i}}\hat{S}^{(4)}(\vec{p})\,, \tag{3.17}$$

where $c_{p_i} \doteq c(\frac{p_i^2}{\Lambda^2})$ and the invariance of $S_0^{(4)}(\vec{p})$ under permutation of the p_i's (which it has without loss of generality) has been utilised. Specialising the above equation to $\vec{p} = (0,0,q,-q)$, eq. (3.16) becomes

$$\begin{aligned}\Lambda\partial_\Lambda S_1^{(2)}(0) &= \frac{1}{\Lambda^2}\int_q c'_q S_0^{(4)}(0,0,q,-q) - \int_q \frac{c_q}{2q^2}\Lambda\partial_\Lambda S_0^{(4)}(0,0,q,-q)\\ &= -\int_q \frac{1}{2q^2}\Lambda\partial_\Lambda\Big(c_q\, S_0^{(4)}(0,0,q,-q)\Big)\\ &= -\Lambda\partial_\Lambda \int_q \frac{c_q\, S_0^{(4)}(0,0,q,-q)}{2q^2}\,.\end{aligned} \tag{3.18}$$

In the above, the derivative with respect to the cutoff may be taken after integrating over the loop momentum since the integral is regulated both in the ultraviolet and in the infrared as a result of the properties of the effective couplings. Eq. (3.18) may be now integrated to give

$$S_1^{(2)}(0) = -\int_q \frac{c_q\, S_0^{(4)}(0,0,q,-q)}{2q^2}\,, \tag{3.19}$$

with no integration constant since for a massless theory, there must be no other explicit scale in the theory apart from the effective cutoff.

Let us now move on to the tree-level six-point function. From (3.7) we get

$$\begin{aligned}\Lambda\partial_\Lambda S_0^{(6)}(\vec{0},q,-q) &= \frac{4q^2}{\Lambda^2}\frac{c'_q}{c_q}\hat{S}^{(6)}(\vec{0},q,-q) -\\ &-\frac{8c'_0}{\Lambda^2}\left[1-\hat{S}^{(4)}(\vec{0})\right]S_0^{(4)}(0,0,q,-q) + \frac{8c'_0}{\Lambda^2}\hat{S}^{(4)}(0,0,q,-q) -\\ &-\frac{12}{\Lambda^2}c'_q\, S_0^{(4)}(0,0,q,-q)\left[S_0^{(4)} - 2\hat{S}^{(4)}\right](0,0,q,-q)\,.\end{aligned} \tag{3.20}$$

Using eq. (3.17), and solving for $\hat{S}^{(6)}(\vec{0},q,-q)$,

$$\begin{aligned}\hat{S}^{(6)}(\vec{0},q,-q) &= \frac{\Lambda^2}{4q^2}\frac{c_q}{c'_q}\bigg\{\Lambda\partial_\Lambda S_0^{(6)}(\vec{0},q,-q)) + \frac{8c'_0}{\Lambda^2}\left[1-\hat{S}^{(4)}(\vec{0})\right]S_0^{(4)}(0,0,q,-q) -\\ &\quad - 2c'_0\frac{c_q}{q^2 c'_q}\Lambda\partial_\Lambda S_0^{(4)}(0,0,q,-q) -\\ &\quad - \frac{6}{q^2}S_0^{(4)}(0,0,q,-q)\Lambda\partial_\Lambda\left[c_q\, S_0^{(4)}(0,0,q,-q)\right]\bigg\}.\end{aligned} \tag{3.21}$$

We will see that substituting eqs. (3.19) and (3.21) into eq. (3.12) will cause almost all the non universal terms to cancel out. The remaining ones will disappear once γ_1 is substituted using eq. (3.13), leaving just the precise form of the one-loop beta function.

Note that in eq. (3.21) and later, it appears at first sight that we need to be able to take the inverse $1/c'_q$. This would mean that in addition to the general restrictions on $\hat{S}$ outlined earlier (and in the conclusions) we would also require that c' does not vanish at finite argument. In fact, we could arrange the calculation more carefully so that $1/c'$ never appears, thus e.g. here we can recognize that only $c'_q \hat{S}^{(6)}(\vec{0}, q, -q)$ is needed for eq. (3.12) and that from eq. (3.17), $\Lambda\partial_\Lambda S_0^{(4)}(0,0,q,-q)$ has a factor of c'_q. For clarity's sake, we will continue to write $1/c'$ in intermediate results and leave as an exercise for the reader to check that all such inverses can be eliminated.

Returning to the calculation in detail, the first term in (3.21) and the $S_0^{(6)}$ term in (3.12) may be paired up into

$$\Lambda\partial_\Lambda \int_q \frac{c_q}{2q^2}\, S_0^{(6)}(\vec{0}, q, -q)\,, \tag{3.22}$$

where again, due to the properties of the effective action vertices, the order of the derivative and integral signs can be exchanged. Moreover, as the integrand in eq. (3.22) is dimensionless, there cannot be any dependence upon Λ after the momentum integral has been carried out, hence the result vanishes identically! Also, the second term in (3.21), when substituted into (3.12), exactly cancels the first term of the latter once (3.19) is used. One is then left with

$$\begin{aligned}\beta_1 + 2\gamma_1 &= -c'_0 \int_q \frac{c_q^2}{q^4 c'_q} \Lambda\partial_\Lambda S_0^{(4)}(0,0,q,-q) - 3\int_q \frac{c_q}{q^4} S_0^{(4)}(0,0,q,-q)\Lambda\partial_\Lambda\left\{c_q S_0^{(4)}(0,0,q,-q)\right\} \\ &= -c'_0 \int_q \frac{c_q^2}{q^4 c'_q} \Lambda\partial_\Lambda S_0^{(4)}(0,0,q,-q) - \frac{3}{2}\int_q \frac{1}{q^4}\Lambda\partial_\Lambda\left\{c_q\, S_0^{(4)}(0,0,q,-q)\right\}^2 . \end{aligned} \tag{3.23}$$

In order to cancel out the first term in eq. (3.23), the one-loop contribution of the wave function renormalization coming from eq. (3.13) must be taken into account. Again making use of eq. (3.17) to rid us of the hatted four-point coupling,

$$\frac{1}{\Lambda^2}\,\hat{S}^{(4)}(p,-p,q,-q)\Big|_{p^2} = \frac{c_q}{4q^2 c'_q}\Lambda\partial_\Lambda S_0^{(4)}(p,-p,q,-q)\Big|_{p^2} - c'_0\left(\frac{c_q}{2q^2 c'_q}\right)^2 \Lambda\partial_\Lambda S_0^{(4)}(0,0,q,-q)\,, \tag{3.24}$$

and substituting back in eq. (3.13),

$$\beta_1 + \gamma_1 = \frac{1}{2}\Lambda\partial_\Lambda \int_q c_q\, S_0^{(4)}(p,-p,q,-q)\Big|_{p^2} - \frac{c'_0}{2}\int_q c'_q \left(\frac{c_q}{q^2 c'_q}\right)^2 \Lambda\partial_\Lambda S_0^{(4)}(0,0,q,-q)\,. \tag{3.25}$$

The first term on the right hand side of eq. (3.25) vanishes as it is a dimensionless UV and IR convergent integral, and therefore γ_1 takes the form

$$\gamma_1 = -\beta_1 - \frac{c'_0}{2}\int_q c'_q \left(\frac{c_q}{q^2 c'_q}\right)^2 \Lambda\partial_\Lambda S_0^{(4)}(0,0,q,-q)\,. \tag{3.26}$$

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Finally, substituting (3.26) in (3.23) yields

$$\begin{aligned}\beta_1 &= \frac{3}{2}\int_q \frac{1}{q^4}\Lambda\partial_\Lambda\Big\{c_q S_0^{(4)}(0,0,q,-q)\Big\}^2 \\ &= -\frac{3}{2}\frac{\Omega_4}{(2\pi)^4}\int_0^\infty dq\,\partial_q\,\Big\{c_q S_0^{(4)}(0,0,q,-q)\Big\}^2 \\ &= \frac{3}{16\pi^2}\,, \end{aligned}\tag{3.27}$$

which is the standard one-loop result [19].[4] Note that in the top equation the Λ derivative cannot be taken outside the integral, as this latter would not then be properly regulated in the infrared. Moreover, had that been possible, it would have resulted in a vanishing beta function, as the integral is actually dimensionless.

4. Summary and conclusions

Starting with the generalised exact RG flow equation (3.2), we computed tree level two, four and six point vertices. At one-loop we computed the effective mass $S_1^{(2)}(0)$ and wavefunction renormalization γ_1. By combining all these with the flow of the one-loop four-point vertex at zero momentum, we arrived at eq. (3.27), which collapses to the expected universal result $\beta_1 = 3/(4\pi)^2$.

The flow equation we used differs from the Polchinski flow equation (2.4), equivalently eq. (2.7), because the seed action $\hat{S}$ is no longer set to be just the kinetic term (2.6), but is generalised to include all arbitrary even higher-point vertices. These are subject only to some very weak and generic restrictions which are recalled below.

In addition we added the anomalous dimension γ term in eq. (3.2) to take account of wavefunction renormalization. Normally this is needed only from two loops onwards, but the more general field reparametrisation induced by the generalised $\hat{S}$, means that a wavefunction renormalization Z is required for the effective action S even at one loop. (The γ term does not exactly follow from the flow equation (2.7) with cutoff c, but rather starts from one with cutoff Zc, but this is more appropriate for cases where wavefunction renormalization is involved.) As a final modification, we also scaled the coupling λ out of the bare action, by rescaling the field. However the result (3.6), is still an equivalent flow to (3.2), since they are related by this simple change of variables. (In particular this means that the higher point vertices in $\hat{S}$ are those of eq. (3.2) multiplied by powers of λ.) Perturbative expansion in λ is now at its most elegant since it coincides with expanding in $\hbar$ i.e. the loop expansion. The structure also most closely coincides with the one used for gauge theory [10, 14, 16], so it acts as an independent test of this part of that framework.[5] We then proceeded to compute the tree and one-loop

[4]The term in braces depends only on q^2/Λ^2. Ω_4 is the four dimensional solid angle. The last line follows from the convergence of the integral and normalisation conditions $c(0) = 1$ and (3.10). As far as independence with respect to the choice of cutoff function is concerned, this is standard.

[5]Much less general unpublished tests were undertaken as preparation for the research in refs. [10, 14].

corrections exploiting the ability, within the exact RG, to derive directly the renormalized couplings and vertices (i.e. without having to refer back to an overall cutoff and bare action).

The effective action of the Polchinski flow equation, eq. (2.4), can be shown to be essentially the generating function for connected diagrams in a field theory with infrared cutoff [2]. As a result the quantum corrections to this effective action have a straightforward interpretation in terms of simple modifications of the usual Feynman diagrams that follow from the original partition function [2]. Since this direct link is obscured by the further field redefinitions implied by the generalised $\hat{S}$ [13], we no longer expect the diagrammatic interpretation of the quantum corrections to be quite so simple. This expectation is indeed borne out by many of the equations we presented, such as eqs. (3.12), (3.13), (3.20) etc. Remarkably however, once $\hat{S}$ vertices are eliminated in favour of those of the wilsonian effective action S, two of our results do have such simple interpretations. One of these is eq. (3.19) which is the one-loop effective mass term required to ensure that the theory is massless once all of the one-loop calculation is completed (i.e. once $\Lambda \to 0$). One can see that it is nothing but the usual tadpole term, formed from the classical effective four point coupling $S_0^{(4)}(\vec{q};\Lambda)$ and the regularised propagator c_q/q^2. Note that the result of the integral is *not* universal: it depends on the details of c, $S_0^{(4)}$ and thus $\hat{S}$ etc., but the form it takes is invariant under scheme changes. It is therefore in this sense, *scheme covariant*. Another scheme covariant expression, and again with a simple diagrammatic interpretation, is eq. (3.27). This is nothing but the standard one-loop diagrammatic result for the β function, again written in terms of c_q/q^2 and the effective $S_0^{(4)}$. As we saw, from here it is straightforward to recognize that the result is universal depending only on the normalisation requirement $c(0) = 1$ and the renormalization condition $S^{(4)}(\vec{0};\Lambda) = 1$, which together with eq. (3.9), define what we mean by the renormalized field and coupling λ, respectively.

We could now argue that we should have expected these results, without the detailed calculation. Nevertheless this is the first specific test of these ideas beyond that of just cutoff function independence, and in the process we found the restrictions on $\hat{S}$ sufficient to ensure scheme independence at this level. They are merely that the seed vertices be infinitely differentiable and lead to convergent momentum integrals. These conditions are needed in any case, because the first condition ensures that no spurious infrared singularities are introduced [10, 3], and the second condition is necessary for the flow equation to make sense at the quantum level and also ensures that the flow actually corresponds to integrating out modes [14, 13].

Finally, a practical method for computing with these generalised exact RGs has been developed. In this respect, it is important to stress that many of our specific choices (what we chose to generalise in Ψ, how we incorporated wavefunction renormalization, organised and solved the perturbative expansion) are not crucial to the calculation. On the contrary there are very many ways to organise the computation; we just chose our favourite one. The crucial step in navigating the generalised corrections, appears to be the recognition that one should eliminate the elements put in by hand, in this case vertices of $\hat{S}$, in favour of the induced solution: the wilsonian effective action S. Indeed our computation just amounts to using this procedure several times over, after which many terms are found to cancel and we are left with particularly simple manifestly scheme covariant results, from which we can recover the expected scheme independent final results. Intuitively, this makes sense, since what are merely our choices are encoded in Ψ (here $\hat{S}$), whilst the actual physics is encoded in S.

For us, this is the most important conclusion of the present paper since it implies a practical prescription for streamlined calculations which can be used even in more involved settings such as in the manifestly gauge invariant framework [10, 14, 16], where there is no equivalent calculation one can directly compare to.

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